Polynomial System Example Bank

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1 Introduction

This file contains a collection of examples for use in evaluating algorithms related to cylindrical algebraic decomposition.

This file is stored permanently at `http://www.cs.bath.ac.uk/~djw42/triangular/examplebank.pdf`. This file is intended as a reference file for the Maple file (stored at `http://www.cs.bath.ac.uk/~djw42/triangular/examplebank.txt`) and the Qepcad file (stored at `http://www.cs.bath.ac.uk/~djw42/triangular/QEPCADexamplebank.txt`).

Each example is given as a Tarski formula or list of polynomials followed by a list of free variables, a list of quantified variables, a suggested variable order (if any), the number of cells in a minimally achievable full CAD (with details of how to reproduce), notes on the problem, and the source.

1.1 Paper-specific Details

The following examples have been used in the stated papers:

- "Speeding up Cylindrical Algebraic Decomposition by Gröbner Bases" by D. J. Wilson, R. J. Bradford and J. H. Davenport:
  - From Section 2: 2, 4, 6–8, 13–14;
  - From Section 5: 1–5;
  - From Section 6: 12–13.

Note that Example 5.4 and Example 2.15 are actually reformulations of the same Solotareff problem. An explanation for this is given at `http://www.cs.bath.ac.uk/~djw42/triangular/solotareff3.pdf`.

2 Examples from [CMXY09]

2.1 Parametric Parabola

$$(\exists\ x)\ [ax^2 + bx + c = 0] \tag{1}$$

Free Variables: a, b, c.
Quantified Variables: x.
Suggested variable order: $x > c > b > a$.
Best achievable number of cells:
Notes:
Source: [CMXY09]

2.2 Whitney Umbrella

$$(\exists\ u)(\exists\ v)\ [x - uv = 0 \wedge y - v = 0 \wedge z - u^2 = 0] \tag{2}$$

Free Variables: x, y, z.
Quantified Variables: u, v.
Suggested variable order: $v > u > z > y > x$.
Best achievable number of cells:
Notes:
Source: [CMXY09]

2.3 Quartic

$$(\forall\ x)\ [x^4 + px^2 + qx + r \geq 0] \tag{3}$$

Free Variables: p, q, r.
Quantified Variables: x.
Suggested variable order: $x > p > q > r$
Best achievable number of cells:
Notes:
Source: [Laz88]

2.4 Sphere and Catastrophe

$$z^2 + y^2 + x^2 - 1 = 0 \quad z^3 + xz + y = 0 \tag{4}$$

Free Variables: x, y, z.
Quantified Variables:
Suggested variable order: $x > y > z$
Best achievable number of cells:
Notes: Full CAD
Source: [McC88]

2.5 Arnon-84

$$y^4 - 2y^3 + y^2 - 3x^2y + 2x^4 = 0 \quad (5)$$

Free Variables: x, y.
Quantified Variables:
Suggested variable order: $y > x$.
Best achievable number of cells:
Notes:
Source: [CMA82]

2.6 Arnon-84-2

$$144y^2 + 96x^2y + 9x^4 + 105x^2 + 70x - 98 = 0 \quad (6)$$

$$xy^2 + 6xy + x^3 + 9x = 0 \quad (7)$$

Free Variables: x, y.
Quantified Variables:
Suggested variable order: $y > x$.
Best achievable number of cells:
Notes:
Source: [CMA82]

2.7 A Real Implicitization Problem

$$(\exists\ u)(\exists\ v)\left[x - uv = 0\ \wedge\ y - uv^2 = 0\ \wedge\ z - u^2 = 0\right] \quad (8)$$

Free Variables: x, y, z.
Quantified Variables: u, v.
Suggested variable order: $v > u > z > y > x$.
Best achievable number of cells:
Notes:
Source: [DSS04]

2.8 Ball and Circular Cylinder

$$(\exists\ z)(\exists\ x)(\exists\ y)\left[[x^2 + y^2 + z^2 - 1 < 0]\ \wedge\ [x^2 + (y + z - 2)^2 - 1 < 0]\right] \quad (9)$$

Free Variables:
Quantified Variables: x, y, z.
Suggested variable order: $z > x > y$
Best achievable number of cells:
Notes: Decides whether the intersection of the open ball with radius 1 centered at the origin and the open circular cylinder with radius 1 and axis the line $x = 0$, $y + z = 2$ is nonempty.
Source: [McC88]

2.9 Termination of Term Rewrite System

$$(\exists\ r)(\forall\ x)(\forall\ y)\Big[[x - r > 0]\ \wedge\ [y - r > 0] \longrightarrow\ [x^2(1 + 2y)^2 - y^2(1 + 2x^2) > 0]\Big] \quad (10)$$

Free Variables:
Quantified Variables: r, x, y.
Suggested variable order: $r > x > y$.
Best achievable number of cells:
Notes: Decides whether we should orient the equation $(xy)^{-1} = y^{-1}x^{-1}$ into $(xy)^{-1} \to y^{-1}x^{-1}$ in order to get a terminating rewrite system for group theory. It uses a polynomial interpretation: $xy \Rightarrow x + 2xy$, $x^{-1} \Rightarrow x^2$, and $1 \Rightarrow 2$.
Source: [CH91]

2.10 Collins and Johnson

$$(\exists\ r)\ \Big[[3a^2r + 3b^2 - 2ar - a^2 - b^2 < 0] \wedge\ [3a^2r + 3b^2r - 4ar + r - 2a^2 - 2b^2 + 2a > 0] \wedge\ [a - \frac{1}{2} \geq 0]\ \wedge\ [b > 0]\ \wedge\ [r > 0]\ \wedge\ [r - 1 < 0]\Big] \quad (11)$$

Free Variables: a, b.
Quantified Variables: r.
Suggested variable order: $r > a > b$.
Best achievable number of cells:
Notes: Necessary and sufficient conditions on the complex conjugate roots $a \pm bi$ so that there exists a cubic polynomial with a single real root r in $(0, 1)$ yet more than one variation is obtained.
Source: [CH91]

2.11 Range of Lower Bounds

$$(\forall\ x)(\forall\ a)(\forall\ b)(\forall\ c)(\exists\ z)\Big[\ [(a > 0)\ \wedge\ (az^2 + bz + c \neq 0)] \longrightarrow\ \big[y < ax^2 + bx + c\big]\ \Big] \quad (12)$$

Free Variables: y.
Quantified Variables: a, b, c, x, z.
Suggested variable order:
Best achievable number of cells:
Notes:
Source: [DSS04]

2.12 X-axis Ellipse Problem

$$ab \neq 0 \wedge$$
$$(\forall\, x)(\forall\, y)\Big[\,[b^2(x-c)^2 + a^2y^2 - a^2b^2 = 0] \;\longrightarrow\; [x^2 + y^2 - 1 \leq 0]\,\Big] \quad (13)$$

Free Variables: a, b, c.
Quantified Variables: x, y.
Suggested variable order: $y > x > b > c > a$.
Best achievable number of cells:
Notes:
Source: [DSS04]

2.13 Davenport and Heintz

$$(\exists\, c)(\forall\, b)(\forall\, a)\ \Big[[a - d = 0 \;\wedge\; b - c = 0] \;\vee\; [a - c = 0 \;\wedge\; b - 1 = 0] \;\longrightarrow\; a^2 - b = 0\Big] \quad (14)$$

Free Variables: d.
Quantified Variables: a, b, c.
Suggested variable order: $a > b > c > d$
Best achievable number of cells:
Notes: A special case of a more general formula.
Source: [DH88],[CH91]

2.14 Hong-90

$$(\exists\, a)(\exists\, b)\Big[\; [r + s + t = 0] \;\wedge\; [rs + st + tr - a = 0] \;\wedge\; [rst - b = 0] \;\Big] \quad (15)$$

Free Variables: r, s, t.
Quantified Variables: a, b.
Suggested variable order: $b > a > t > s > r$.
Best achievable number of cells:
Notes:
Source: [Hon90]

2.15 Solotareff-3

$$(\exists\, u)(\exists\, v)\ \Big[[r > 0] \;\wedge\; [r - 1 > 0] \;\wedge\; [u + 1 > 0] \;\wedge\; [u - v < 0] \;\wedge\; [v - 1 < 0] \;\wedge\; [3u^2 + 2ru - a = 0] \;\wedge\; [3v^2 + 2rv - a = 0] \;\wedge\; [u^3 + ru^2 - au + a - r - 1 = 0] \;\wedge\; [v^3 + rv^2 - av - 2b - a + r + 1 = 0]\Big] \quad (16)$$

Free Variables: a, b, r, u, v.
Quantified Variables:
Suggested variable order: $b > u > v > r > a$.
Notes: A reformulation of Solotareff's problem using the outline in [Ach56].
Source: [CMXY09], [Ach56]

2.16 Collision Problem

$$\begin{aligned}(\exists\ t)(\exists\ x)(\exists\ y)\Big[[\frac{17}{16}t - 6 \geq 0]\ \wedge\ [\frac{17}{16}t - 10 \leq 0] \\ \wedge [x - \frac{17}{16}t + 1 \geq 0]\ \wedge\ [x - \frac{17}{16}t - 1 \leq 0] \\ \wedge\ [y - \frac{17}{16}t + 9 \geq 0]\ \ \wedge\ [y - \frac{17}{16}t + 7 \leq 0] \\ \wedge\ [(x-t)^2 + y^2 - 1 \leq 0]\Big]\end{aligned} \quad (17)$$

Free Variables:
Quantified Variables: t, x, y.
Suggested variable order: $t > x > y$.
Notes: Collision of two semi-algebraic objects: a circle with radius 1, initially centered at $(0,0)$ and moving with velocity $v_x = 1$ and $v_y = 0$; a square with side-length 2, initially centered at $(0,-8)$ and moving with velocity $v_x = \frac{17}{16}$ and $v_Y = \frac{17}{16}$.
Source: [CH91]

2.17 McCallum Trivariate Random Polynomial

$$(y-1)z^4 + xz^3 + x(1-y)z^2 + (y-x-1)z + y \quad (18)$$

Free Variables: x, y, z.
Quantified Variables:
Suggested variable order: $z > y > x$.
Best achievable number of cells:
Notes:
Source: [McC88]

2.18 Ellipse Problem

$$\begin{aligned}[ab \neq 0]\ \wedge\ (\forall\ x)(\forall\ y)\Big[[b^2(x-c)^2 + a^2(y-d)^2 - a^2b^2 = 0] \\ \longrightarrow\ [x^2 + y^2 - 1 \leq 0]\Big]\end{aligned} \quad (19)$$

Free Variables: a, b, c, d.
Quantified Variables: x, y.
Suggested variable order: $y > x > d > c > b > a$.
Best achievable number of cells:
Notes:
Source: [CMXY09]

3 Branch Cut Examples

3.1 Square Root Identity

Identity in question: $\sqrt{z-1}\sqrt{z+1} \stackrel{?}{=} \sqrt{z^2-1}$

$$\{[x-1<0] \ \wedge \ [y=0]\} \tag{20}$$

$$\{[x+1<0] \ \wedge \ [y=0]\} \tag{21}$$

$$\left\{[x^2-y^2-1<0] \ \wedge \ [xy=0]\right\}. \tag{22}$$

Free Variables: x, y.
Quantified Variables:
Suggested Variable Order: $x > y$.
Best achievable number of cells:
Notes: Branch cuts for the three square-roots. Input into QEPCAD should be with $\vee$ between each set.
Source: [Phi11]

3.2 Arctan Identity

Identity in question: $\arctan(x) + \arctan(y) \stackrel{?}{=} \arctan\left(\frac{x+y}{1-xy}\right)$

4 Motion Planning Examples

4.1 Piano Mover's Problem (Davenport)

$$\begin{aligned}
\Big[\ &[(x-x')^2+(y-y')^2-9=0] \ \wedge \\
&[[yy' \geq 0] \ \vee \ [x(y-y')^2+y(x'-x)(y-y') \geq 0]] \ \wedge \\
&[[(y-1)(y'-1) \geq 0] \ \vee \ [(x+1)(y-y')^2+(y-1)(x'-x)(y-y') \geq 0]] \ \wedge \\
&[[xx' \geq 0] \ \vee \ [y(x-x')^2+x(y'-y)(x-x') \geq 0]] \ \wedge \\
&[[(x+1)(x'+1) \geq 0] \ \vee \ [(y-1)(x-x')^2+(x+1)(y'-y)(x-x') \geq 0]]\ \Big].
\end{aligned} \tag{23}$$

Free Variables: x, x', y, y'.
Quantified Variables:
Suggested Variable Order:
Best achievable number of cells:
Notes:
Source: [Dav86]

5 Examples from Buchberger–Hong [BH91]

5.1 Intersection

$$(\exists\ z)\Big[[x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 = 0] \ \wedge\ [xz + zy - 2x = 0] \ \wedge\ [z^2 - y = 0]\Big]. \tag{24}$$

Quantified Variables: z.
Free Variables: x, y.
Suggested variable order:
Minimal achieved number of cells:
Notes:
Source: [BH91]

5.2 Random

$$(\exists\ x)(\forall\ y)(\exists\ z)\Big[[4x^2 + xy^2 - z + \frac{1}{4} = 0] \ \wedge \\ [2x + y^2z + \frac{1}{2} = 0] \ \wedge\ [x^2z - \frac{1}{2}x - y^2 = 0]\Big]. \tag{25}$$

Quantified Variables: x, y, z.
Free Variables:
Suggested variable order:
Minimal achieved number of cells:
Notes:
Source: [BH91]

5.3 Ellipse

$$(\exists\ x)(\exists\ y)\Big[[x^2 + y^2 - 1 = 0] \ \wedge\ [b^2(x-c)^2 + a^2y^2 - a^2b^2 = 0] \\ \wedge\ [a > 0] \ \wedge\ [a < 1] \ \wedge\ [b > 0] \ \wedge\ [b < 1] \ \wedge\ [c \geq 0] \ \wedge\ [c < 1]\Big]. \tag{26}$$

Quantified Variables: x, y.
Free Variables: a, b, c.
Suggester variable order:
Minimal achieved number of cells:
Notes: Inspired by Kahan's problem. Instead of having the ellipse contained in the circle, this looks for an intersection.
Source: [BH91]

5.4 Solotareff

$$(\exists\ x)(\exists y)\Big[[3x^2 - 2x - a = 0] \ \wedge\ [x^3 - x^2 - ax - 2b + a - 2 = 0] \\ \wedge\ [3y^2 - 2y - a = 0] \ \wedge\ [y^3 - y^2 - ay - a + 2 = 0] \ \wedge\ [1 \leq 4a] \ \wedge\ [4a \leq 7] \\ \wedge\ [-3 \leq 4b] \ \wedge\ [4b \leq 3] \ \wedge\ [-1 \leq x] \ \wedge\ [x \leq 0] \ \wedge\ [0 \leq y] \ \wedge\ [y \leq 1]\Big]. \tag{27}$$

Quantified Variables: x, y.
Free Variables: a, b.
Suggested variable order:
Minimal achieved number of cells:
Notes:
Source: [BH91]

5.5 Collision

$$(\exists\ t)(\exists\ x)(\exists\ y)\Big[[\frac{1}{4}(x-t)^2-(y-10)^2-1=0] \\ \wedge\ [\frac{1}{4}(x-at)^2+(y-at)^2-1=0]\ \wedge\ [t>0]\ \wedge\ [a>0]\Big]. \quad (28)$$

Quantified Variables: t, x, y.
Free Variables: a.
Suggested variable order:
Minimal achieved number of cells:
Notes: In this problem t stands for time and the problem describes two ellipses with semi-axes 2 and 1. One is centered at $(0, 10)$ moving with horizontal velocity 1. The other is centered at the origin and moving with velocity (a, a). The problem decides if the ellipses collide.
Source: [BH91]

6 Other Examples

6.1 Off-Center Ellipse

$$[a \neq 0] \ \wedge \ (\forall\ x)(\forall\ y)\Big[[16a^2y^2 - 8a^2y + 4x^2 - 4x - 3a^2 + 1 = 0] \longrightarrow \ [y^2 + x^2 - 1 \leq 0]\Big] \tag{29}$$

Free Variables: a.
Quantified Variables: x, y.
Suggested variable order:
Best achievable number of cells:
Notes: Deciding if an off-center ellipse with center $(\frac{1}{2}, \frac{1}{4})$, semi–major axis a and semi–minor axis $\frac{1}{2}$ lies within the unit circle centered at the origin.
Source: [AM88]

6.2 Concentric Circles

$$x^2 + y^2 - 9 = 0 \tag{30}$$

$$x^2 + y^2 - 1 = 0 \tag{31}$$

Free Variables: x, y.
Quantified Variables:
Suggested variable order: $y > x$.
Best achievable number of cells: 41
Notes:
Source: [Dav11]

6.3 Non-Concentric Circles

$$x^2 + y^2 - 9 = 0 \tag{32}$$

$$x^2 + (y-1)^2 - 1 = 0 \tag{33}$$

Free Variables: x, y.
Quantified Variables:
Suggested variable order: $y > x$.
Best achievable number of cells: 41
Notes: Note that an extra spurious point is added compared to the concentric case - corresponding to the "complex intersection" of the circles.
Source: [Dav11]

6.4 Edges Square Product

$$(\exists\ x_1)(\exists\ x_2)(\exists\ y_2)\ \Big[\ [x = x_1x_2 - y_2] \ \wedge \ [y = x_1y_2 + x + 2] \ \wedge \ [0 \leq x_1] \ \wedge \ [x_1 \leq 2] \ \wedge \ [2 \leq x_2] \ \wedge \ [x_2 \leq 4] \ \wedge \ [-1 \leq y_2] \ \wedge \ [y_2 \leq 1] \ \wedge \ [-1 \leq x] \ \wedge \ [x \leq 9] \ \wedge \ [-6 \leq y] \ \wedge \ [y \leq 6]\ \Big] \tag{34}$$

Free Variables: x, y
Quantified Variables:x_1, x_2, y_2
Suggested variable order:
Best achievable number of cells:
Notes: Originally stated by Collins. Solution given in Figure 1.
Source: [BG06]

6.5 Simplified Edges Square Product

$$\begin{aligned}(\exists\, x_1)\Big[\ &[-x_1 \leq 0] \ \wedge\ [x_1 \leq 2] \ \wedge [0 \leq 1 + x] \ \wedge\ [x \leq 9] \ \wedge \\ &[0 \leq y + 6] \ \wedge\ [y \leq 6] \ \wedge\ [0 \leq -(x_1^2+1)(-y - x_1 x + 2x_1^2 + 2)] \ \wedge \\ &[(-x + x_1 y)(x_1^2 + 1) \leq (x_1^2+1)^2] \ \wedge\ [x_1^2 + 1 \neq 0] \ \wedge \\ &[0 \leq (x_1^2+1)(x_1^2 + 1 - x + x_1 y)] \ \wedge\ [(x_1^2+1)(y + x_1 x) \leq 4(x_1^2+1)^2] \ \Big] \end{aligned} \tag{35}$$

Free Variables: x, y
Quantified Variables: x_1
Suggested variable order:
Best achievable number of cells:
Notes: Simplified version of the Edges Square Product given in [BG06] using $x_2 = y - x_1 y_2$ and $y_2 = (-x + x_1 y)/(x_1^2 + 1)$. Solution given in Figure 1.
Source: [BG06]

6.6 Putnum Example

$$\begin{aligned}(\exists\, x_1)(\exists\, y_1)(\exists x_2)(\exists y_2)\Big[\ &x_1^2 + y_1^2 - 1 = 0 \ \wedge\ (x_2 - 10)^2 + y_2^2 - 9 = 0 \\ &\wedge\ x = \frac{x_1 + x_2}{2} \ \wedge y = \frac{y_1 + y_2}{2} \ \Big]\end{aligned} \tag{36}$$

Free Variables: x, y.
Quantified Variables: x_1, y_1, x_2, y_2
Suggested variable order:
Best achievable number of cells:
Notes: Problem 2 from the 57th Putnam competition. What points are halfway between the points on the unit circle centered on the origin and a circle with radius 3 centered at $(10, 0)$. Solution shown in blue in Figure 3.
Source: [BG06]

6.7 Simplified Putnum

$$\begin{aligned}(\exists\, x_1)(\exists\, y_2)\Big[\ &(x_1^2 + 4y^2 - 4yy_2 + y_2^2 - 1 = 0) \\ &\wedge\ (4x^2 - 4xx_1 - 40x + x_1^2 + 20x_1 + 91 + y_2^2 = 0) \ \Big]\end{aligned} \tag{37}$$

Free Variables: x, y.
Quantified Variables: x_1, y_2

Suggested variable order:
Best achievable number of cells:
Notes: The Putnum example following simplifications $x_2 := 2x - x_1$ and $y_1 := 2y - y_2$. Solution shown in blue in Figure 3.
Source: [BG06]

6.8 YangXia

$$(\exists\ s)(\exists\ b)(\exists\ c)\Big[\ (a^2h^2 - 4s(s-a)(s-b)(s-c) = 0)\ \wedge\ (2Rh - bc = 0)\ \wedge$$
$$(2s - a - b - c = 0)\ \wedge\ (b > 0)\ \wedge\ (c > 0)\ \wedge\ (R > 0)\ \wedge\ (h > 0)\ \wedge$$
$$(a + b - c > 0)\ \wedge\ (b + c - a > 0)\ \wedge\ (c + a - b > 0)\Big]. \quad (38)$$

Free Variables: a, h, R.
Quantified Variables: s, b, c.
Suggested variable order:
Best achievable number of cells:
Notes:
Source: [BG06]

6.9 Simplified YangXia

$$(\exists\ b)\Big[\ (-\frac{1}{2}b \neq 0)\ \wedge\ (0 < R)\ \wedge\ (0 < b)\ \wedge\ (0 < h)\ \wedge$$
$$(\frac{1}{16}a^2h^2b^4 - \frac{1}{32}a^2b^6 - \frac{1}{8}a^2R^2h^2b^2 - \frac{1}{8}R^2h^2b^4 + \frac{1}{64}b^8 + \frac{1}{64}a^4b^4 + \frac{1}{4}R^4h^4 = 0)\ \wedge$$
$$(0 < -\frac{1}{4}(-ab - b^2 + 2Rh)b)\ \wedge\ (0 < \frac{1}{2}Rhb)\ \wedge$$
$$(0 < \frac{1}{4}(2Rh + ab - b^2)b)\ \wedge\ (0 < \frac{1}{4}(b^2 + 2Rh - ab)b)\ \Big]. \quad (39)$$

Free Variables: a, h, R
Quantified Variables: b.
Suggested variable order: $b > a > h > R$.
Best achievable number of cells:
Notes: Simplified YangXia using $s := \frac{1}{2}(a + b + c)$ and $c := \frac{2Rh}{b}$.
Source: [BG06]

6.10 SEIT Model

$$(\exists\ s)(\exists\ F)(\exists\ J)(\exists\ T)\ \Big[\ [d - ds - b_1Js = 0]\ \wedge\ [vF - (d + r_2)J = 0]\ \wedge$$
$$[b_1J + b_2JT - (d + v + r_1)F + (1 - q)r_2J = 0]\ \wedge\ [-dT + r_1F + qr_2J - b^2TJ = 0]\ \wedge$$
$$[F > 0]\ \wedge\ [J > 0]\ \wedge\ [T > 0]\ \wedge\ [s > 0]\ \wedge\ [b_1 > 0]\ \wedge\ [d > 0]\ \wedge$$
$$[v > 0]\ \wedge\ [r_1 > 0]\ \wedge\ [r_2 > 0]\ \wedge\ [q > 0]\ \wedge\ [b_1 > b_2]\ \Big]. \quad (40)$$

Free Variables: $b_1, b_2, d, q, r_1, r_2, v$.

Quantified Variables: s, F, J, T.

Suggested variable order:

Best achievable number of cells:

Notes: SEIT Model is used in epidemic modeling. This problem asks for the existence of an endemic equilibrium.

Source: [BG06]

6.11 Simplified SEIT Model

$$(\exists\, J)\ \Big[[0 < d]\ \wedge\ [0 < r_1]\ \wedge\ [0 < r_2]\ \wedge\ [0 < q]\ \wedge\ [b_2 < b_1]\ \wedge\ [0 < v]\ \wedge \\ [0 < J]\ \wedge\ [0 < b_1]\ \wedge\ [0 < b_2]\ \wedge\ [d + Jb_1 \neq 0]\ \wedge\ [-v \neq 0]\ \wedge \\ [0 < (d + r_2)Jv]\ \wedge\ [vb_2 \neq 0]\ \wedge\ [0 < d(d + Jb_1)]\ \wedge \\ [0 < (d + Jb_1)b_2v(-dvb_1 + d^2v + d^2r_2 + dvr_2q + d^3 + d^2r_1 + Jb_1vr_2q+ \\ dr_2r_1 + Jb_1dv + Jb_1dr_2 + Jb_1r_2r_1 + Jb_1d^2 + Jb_1dr_1)]\ \wedge \\ [-(d + Jb_1)b_2v^3d(-dvb_1 - Jb_1vb_2 + d^2v + d^2r_1 + dvr_2q + d^3+ \\ b_2Jd^2 + d^2r_2 + dr_2r_1 + b_2Jdv + b_2Jdr_2 + Jb_1dv + Jb_1dr_1 + Jb_1vr_2q+ \\ Jb_1d^2 + J^2b_1db_2 + Jb_1dr_2 + Jb_1r_2r_1 + J^2b_1b_2v + J^2b_1r_2b_2) = 0]\ \Big]. \quad (41)$$

Free Variables: $b_1, b_2, d, q, r_1, r_2, v$.

Quantified Variables: J.

Suggested variable order: $J > r_1 > q > r_2 > b_2 > v > d > b_1$

Best achievable number of cells:

Notes: Simplified using $F := (d + r_2)J/v$, factorization and substitution of $J := 0$, $T := (-vb_1s + dr_1 + vr_2q + d^2 + dv + dr_2 + r_2r_1)/(vb_2)$ and $v := 0$, as well as the substitution $s := d/(d + Jb_1)$ and cutting of all contradicting subformulas.

Source: [BG06]

6.12 Cyclic—3

$$(\exists\, b)(\exists\, c)\ \Big[[a + b + c = 0]\ \wedge\ [ab + bc + ca = 0]\ \wedge\ [abc - 1 = 0]\Big]. \quad (42)$$

Free Variables: a.

Quantified Variables: b, c.

Suggested variable order: $c > b > a$.

Best achievable number of cells:

Notes:

Source:

6.13 Cyclic—4

$$(\exists\, b)(\exists\, c)(\exists\, d)\ \Big[[a + b + c + d = 0]\ \wedge\ [ab + bc + cd + da = 0]\ \wedge \\ [abc + bcd + cda + dab = 0]\ \wedge\ [abcd - 1 = 0]\Big]. \quad (43)$$

Free Variables: a.
Quantified Variables: b, c, d.
Suggested variable order: $c > b > a$.
Best achievable number of cells:
Notes:
Source:

6.14 Kauers11-1

$$x^2 + y^2 - 4 \tag{44}$$
$$(x-1)(y-1) - 1. \tag{45}$$

Free Variables: x, y.
Quantified Variables:
Suggested variable order: $x > y$.
Best achievable number of cells: 59
Notes:
Source: [Kau11]

A Figures

The following are a list of figures to accompany the examples.

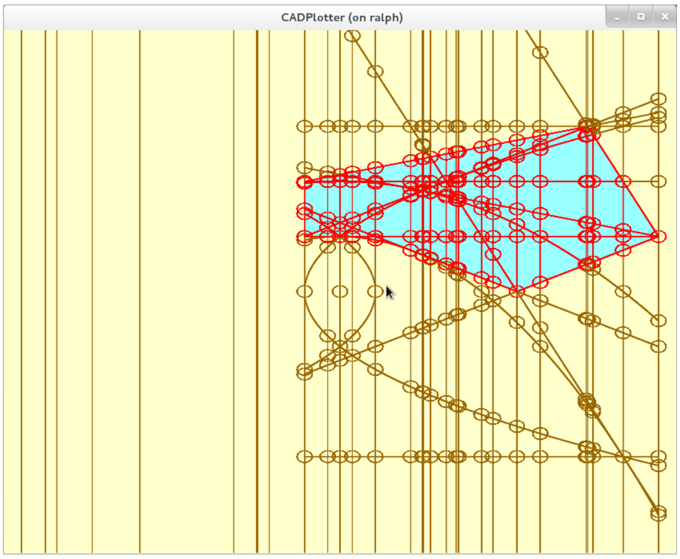

Figure 1: CAD produced for the Edges Square Product problem

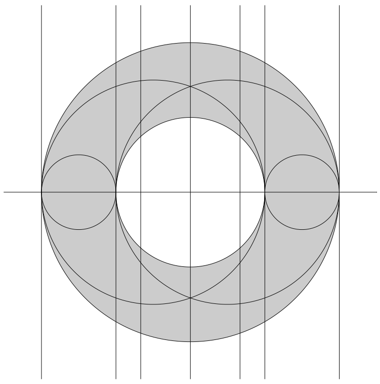

Figure 2: Partial CAD generated from the Simplified Putnum problem

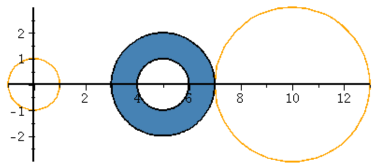

Figure 3: Solution set for the Putnum example, plotted with the original circles.

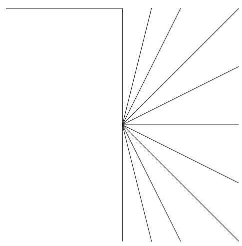

Figure 4: 2D-CAD produced for the simplified YangXia problem (R against h)

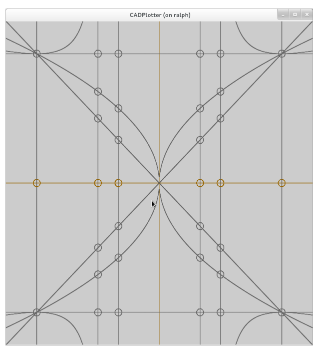

Figure 5: Projected 2D-CAD produced for the X-axis ellipse problem

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