```
> restart; kernelopts(opaquemodules=false):
```

We are using the standard Maple 18 Library

```
> libname:="/home/me350/Programs/Maple18/lib";
```

$$libname := \text{"/home/me350/Programs/Maple18/lib"} \tag{1}$$

We make use of the following packages:

```
> with(plots):
  with(RegularChains):
  with(SemiAlgebraicSetTools):
```

Additionally, we use Maple code written at the University of Bath: The ProjectionCAD package (should be hosted alongside this worksheet).

```
> read("ProjectionCAD.mpl"):
  with(ProjectionCAD):
```

"This is V3.18 of the ProjectionCAD module from 11th February 2015, designed and tested for use in Maple 18." (2)

# Section 4 Example

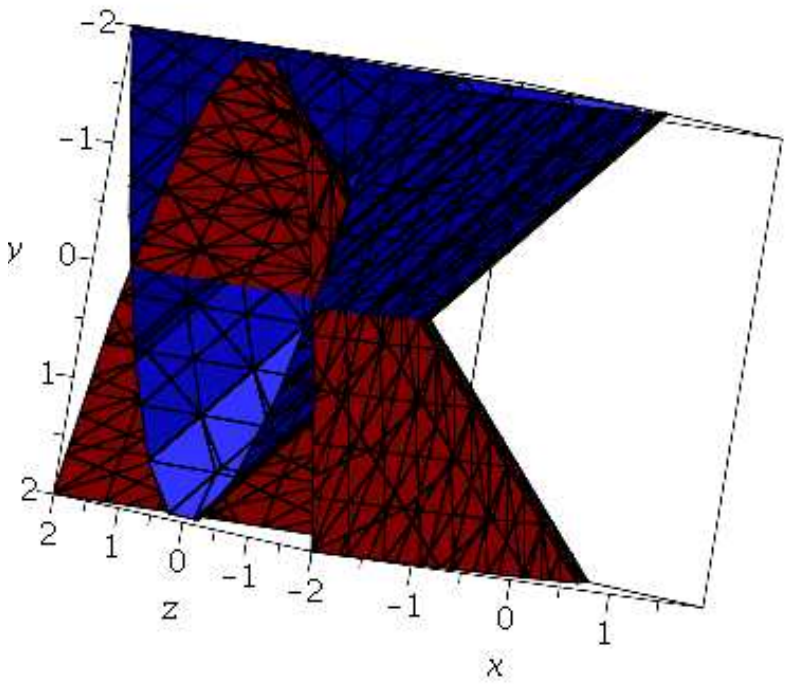
```
> f1 := x-y+z^2:
  f2 := z^2-u^2+v^2-1:
  f3 := x+y+z^2:
  f4 := z^2+u^2-v^2-1:
  g := x^2-1:
  h := z:
  F := [f1,f2,f3,f4,g,h]:
  lsas := [[f1=0, f2=0, f3=0, f4=0, g>=0, h>=0]];
  ord := [z,y,x,u,v]:
  R := PolynomialRing(ord):
```

$$lsas := [[z^2 + x - y = 0, -u^2 + v^2 + z^2 - 1 = 0, z^2 + x + y = 0, u^2 - v^2 + z^2 - 1 = 0, 0 \le x^2 - 1, 0 \le z]] \tag{1.1}$$

## Solution by hand

What is the solution?
Consider f1 and f3 on their own first

```
> implicitplot3d([f1,f3], x=-2..2, y=-2..2, z=-2..2, color=
  [red,blue]);
```

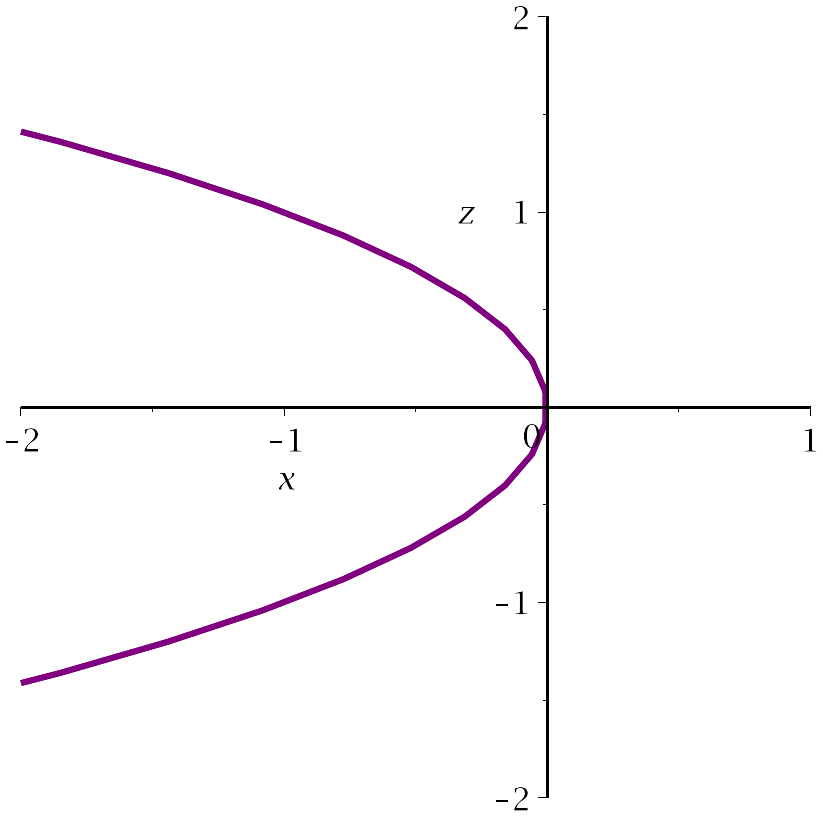



```
> resultant(f1,f3,z);
```

$$4\,y^2 \qquad \textbf{(1.1.1)}$$

```
> subs(y=0,[f1=0,f3=0]);
  plots:-implicitplot(%, x=-2..1, z=-2..2, color=["Purple"],
  thickness=3, view=[-2..1, -2..2]);
```

$$[z^2 + x = 0,\ z^2 + x = 0]$$

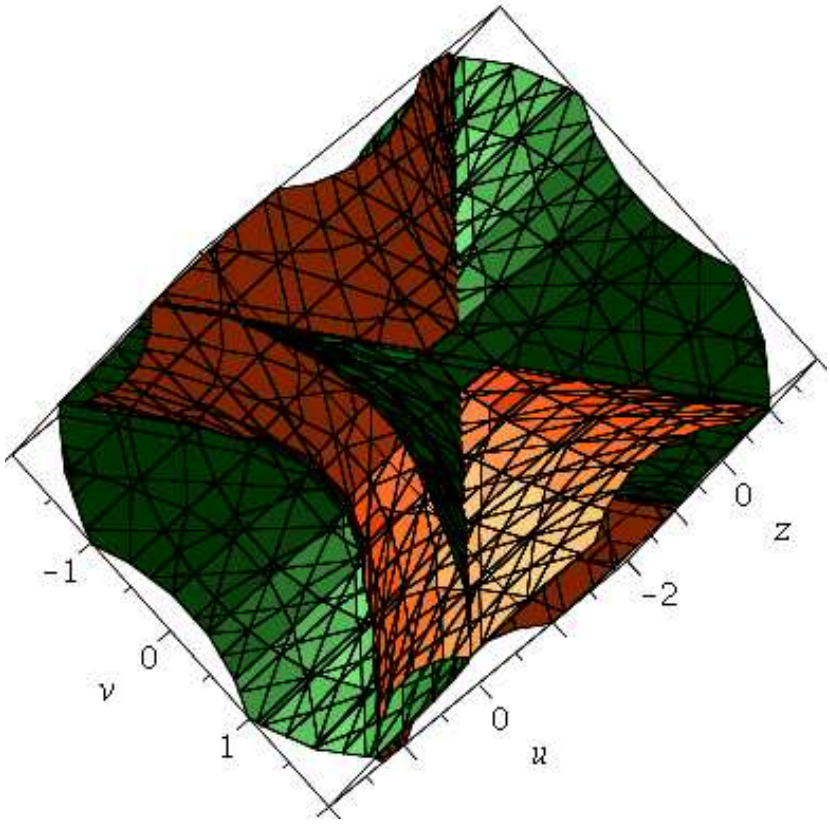



```
> subs(y=0, f1);
  subs(y=0, f3);
```

$$z^2 + x$$

$$z^2 + x \qquad \textbf{(1.1.2)}$$

```
> implicitplot3d([f2,f4], u=-2..2, v=-2..2, z=-2..2, color=
  ["DarkGreen","OrangeRed"]);
```




```
> resultant(f2,f4,z); solve(%);
```

$$(2\,u^2 - 2\,v^2)^2$$

$$\{u = v,\ v = v\},\ \{u = -v,\ v = v\} \qquad \textbf{(1.1.3)}$$

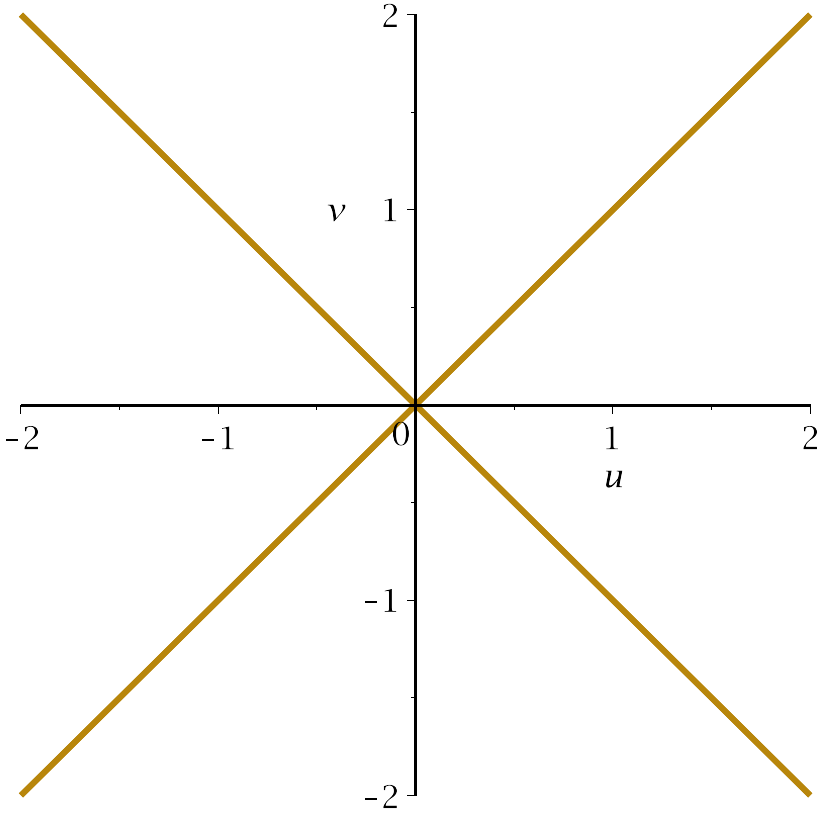
```
> algsubs(u^2=v^2, f2);
  algsubs(u^2=v^2, f4);
```

$$z^2 - 1$$

$$z^2 - 1 \qquad \textbf{(1.1.4)}$$

```
> subs(z=1, [f2,f4]);
  plots:-implicitplot(%, u=-2..2, v=-2..2, color=
  ["DarkGoldenrod"], thickness=3, numpoints=400, view=[-2..2,
  -2..2]);
  #subs(z=-1, [f2,f4]);
  #plots:-implicitplot(%, u=-2..2, v=-2..2, color=["Brown"],
  thickness=3, numpoints=400, view=[-2..2, -2..2]);
```

$$[-u^2 + v^2,\ u^2 - v^2]$$




So we need y=0 and u^2=v^2. To satisfy the ECs together we thus need

```
> algsubs(u^2=v^2, subs(y=0, {f1,f2,f3,f4})); solve(%);
```

$$\{z^2-1,\ z^2+x\}$$

$$\{x=-1,\ z=1\},\ \{x=-1,\ z=-1\} \qquad \textbf{(1.1.5)}$$

z = +/- 1 and x = -1.

```
> algsubs(u^2=v^2, subs(x=-1,y=0, [f1,f2,f3,f4] )): convert(%,
  set);
```

$$\{z^2-1\} \qquad \textbf{(1.1.6)}$$

But z must be positive from h and so only z=+1 allowed. The other non-EC, g, is satisfied by the x coordinate.
Two solution sets:

```
> Solution := { [x=-1, y=0, z=1, u=v],  [x=-1, y=0, z=1, u=-v]
  };
```

$$Solution := \{[x=-1,\ y=0,\ z=1,\ u=v],\ [x=-1,\ y=0,\ z=1,\ u=-v]\} \qquad \textbf{(1.1.7)}$$

## What ECs are there?

We work systematically. In all 5 variables we have

```
> f1;f2;f2;f3;
```

$$z^2+x-y$$
$$-u^2+v^2+z^2-1$$
$$-u^2+v^2+z^2-1$$
$$z^2+x+y \qquad (1.2.1)$$

In 4 variables we have

```
> [resultant(f1,f2,z),
  resultant(f1,f3,z),
  resultant(f1,f4,z),
  resultant(f2,f3,z),
  resultant(f2,f4,z),
  resultant(f3,f4,z)];
  L2EC:=convert(ProjectionCAD:-PCAD_SFBasis( select(X->has(X,
  y), %) ,x), list);
  L4EC:=convert(ProjectionCAD:-PCAD_SFBasis( remove(X->has(X,
  y), %%), v), list);
```

$$\left[(-u^2+v^2-x+y-1)^2,\ 4y^2,\ (u^2-v^2-x+y-1)^2,\ (u^2-v^2+x+y+1)^2,\ (2u^2-2v^2)^2,\ (u^2-v^2-x-y-1)^2\right]$$

$$L2EC := [4y^2,\ -u^2+v^2+x-y+1,\ -u^2+v^2+x+y+1,\ u^2-v^2+x-y+1,\ u^2-v^2+x+y+1]$$

$$L4EC := [-u^2+v^2] \qquad (1.2.2)$$

```
> nops(L2EC);
```

$$5 \qquad (1.2.3)$$

```
> [
  resultant(L2EC[1],L2EC[2],y),
  resultant(L2EC[1],L2EC[3],y),
  resultant(L2EC[1],L2EC[4],y),
  resultant(L2EC[1],L2EC[5],y),
  resultant(L2EC[2],L2EC[3],y),
  resultant(L2EC[2],L2EC[4],y),
  resultant(L2EC[2],L2EC[5],y),
  resultant(L2EC[3],L2EC[4],y),
  resultant(L2EC[3],L2EC[5],y),
  resultant(L2EC[4],L2EC[5],y)
  ];
  L3EC := convert(ProjectionCAD:-PCAD_SFBasis( select(X->has
  (X,x), %) ,x), list);
  convert(ProjectionCAD:-PCAD_SFBasis( remove(X->has(X,x), %)
  ,v), list);
```

$$\left[4(-u^2+v^2+x+1)^2,\ 4(-u^2+v^2+x+1)^2,\ 4(u^2-v^2+x+1)^2,\ 4(u^2-v^2\right.$$

$$+x+1)^2,\ 2u^2-2v^2-2x-2,\ -2u^2+2v^2,\ -2x-2,\ 2x+2,\ 2u^2-2v^2,\ -2u^2+2v^2-2x-2]$$

$$L3EC := [x+1,\ -u^2+v^2+x+1,\ u^2-v^2+x+1]$$

$$[\,] \qquad \textbf{(1.2.4)}$$

```
> nops(L3EC);
```

$$3 \qquad \textbf{(1.2.5)}$$

```
> [
  resultant(L3EC[1],L3EC[2],x),
  resultant(L3EC[1],L3EC[3],x),
  resultant(L3EC[2],L3EC[3],x)
  ];
  L4EC:=convert(ProjectionCAD:-PCAD_SFBasis( select(X->has(X,
  u), %) ,u), list);
  convert(ProjectionCAD:-PCAD_SFBasis( remove(X->has(X,u), %)
  ,v), list);
```

$$[-u^2+v^2,\ u^2-v^2,\ 2u^2-2v^2]$$

$$L4EC := [u^2-v^2]$$

$$[\,] \qquad \textbf{(1.2.6)}$$

```
> L2EC;
  L3EC;
  L4EC;
```

$$[4y^2,\ -u^2+v^2+x-y+1,\ -u^2+v^2+x+y+1,\ u^2-v^2+x-y+1,\ u^2-v^2+x+y+1]$$

$$[x+1,\ -u^2+v^2+x+1,\ u^2-v^2+x+1]$$

$$[u^2-v^2] \qquad \textbf{(1.2.7)}$$

```
> L1EC := [f1,f2,f3,f4];
  L2EC := [4*y^2, -u^2+v^2+x-y+1, -u^2+v^2+x+y+1, u^2-v^2+x-y+1,
  u^2-v^2+x+y+1];
  L3EC := [x+1, -u^2+v^2+x+1, u^2-v^2+x+1];
  L4EC := [u^2-v^2];
```

$$L1EC := [z^2+x-y,\ -u^2+v^2+z^2-1,\ z^2+x+y,\ u^2-v^2+z^2-1]$$

$$L2EC := [4y^2,\ -u^2+v^2+x-y+1,\ -u^2+v^2+x+y+1,\ u^2-v^2+x-y+1,\ u^2-v^2+x+y+1]$$

$$L3EC := [x+1,\ -u^2+v^2+x+1,\ u^2-v^2+x+1]$$

$$L4EC := [u^2-v^2] \qquad \textbf{(1.2)}$$

# Solutions using new ideas as described in paper

```
> resultant(f1,f2,z);
  r1:=-u^2+v^2-x+y-1;
```

$$(-u^2+v^2-x+y-1)^2$$

$$r1 := -u^2+v^2-x+y-1 \quad \textbf{(1.3.1)}$$

```
> A5:={f1,f2,f3,f4,g,h}: A5:=ProjectionCAD:-PCAD_SFBasis(A5,z)
  ;
  E5:={f1};
```

$$A5 := \{z,\ x^2-1,\ z^2+x-y,\ z^2+x+y,\ -u^2+v^2+z^2-1,\ u^2-v^2+z^2-1\}$$

$$E5 := \{z^2+x-y\} \quad \textbf{(1.3.2)}$$

No need for extra discriminants as first projection

```
> A4 := [ op(ProjectionCAD:-ECCADProjOp( E5, A5, ord)) ]:
  A4 := ProjectionCAD:-PCAD_SFBasis(%, y);
  E4 := [r1];
```

$$A4 := \{y,\ (x^2-1)^2,\ -x+y,\ -u^2+v^2-x+y-1,\ u^2-v^2-x+y-1\}$$

$$E4 := [-u^2+v^2-x+y-1] \quad \textbf{(1.3.3)}$$

This time need to consider extra discrim

```
> A3 := { op(ProjectionCAD:-ECCADProjOp( convert(E4,set),
  convert(A4,set), [y,x,u,v])) };
  convert(map(X->discrim(X,y), expand(A4)),set);
```

$$A3 := \{(x^2-1)^2,\ 2u^2-2v^2,\ u^2-v^2+1,\ u^2-v^2+x+1\}$$

$$\{0,\ 1\} \quad \textbf{(1.3.4)}$$

All evaluate to constants

```
> resultant( resultant(f1,f2,z), resultant(f1,f3,z), y);
  r2:=u^2-v^2+x+1;
```

$$16\,(u^2-v^2+x+1)^4$$

$$r2 := u^2-v^2+x+1 \quad \textbf{(1.3.5)}$$

```
> A3 := ProjectionCAD:-PCAD_SFBasis(A3, x);
  E3:=[r2];
```

$$A3 := \{x^2-1,\ 2u^2-2v^2,\ u^2-v^2+1,\ u^2-v^2+x+1\}$$

$$E3 := [u^2-v^2+x+1] \quad \textbf{(1.3.6)}$$

consider extra discriminants

```
> A2 := { op(ProjectionCAD:-ECCADProjOp( convert(E3,set),
  convert(A3,set), [x,u,v])) };
  convert(map(X->discrim(X,x), A3),set);
```

$$A2 := \{2u^2-2v^2,\ u^2-v^2+1,\ u^4-2u^2v^2+v^4+2u^2-2v^2\}$$

$$\{0,\ 1,\ 4\} \quad \textbf{(1.3.7)}$$

All evaluate to constants.

```
> resultant( RR, resultant(f1,f4,z), y);
  r3:=u^2-v^2;
```

$$(2\,u^2 - 2\,v^2)^2$$

$$r3 := u^2 - v^2 \tag{1.3.8}$$

```
> A2 := ProjectionCAD:-PCAD_SFBasis(A2, u);
  E2:=[u^2-v^2]; factor(%);
```

$$A2 := \{u^2 - v^2,\ u^2 - v^2 + 1,\ u^4 - 2\,u^2\,v^2 + v^4 + 2\,u^2 - 2\,v^2\}$$

$$E2 := [u^2 - v^2]$$

$$[(u - v)\ (u + v)] \tag{1.3.9}$$

No need for discrim in final projection.

```
> A1 := { op(ProjectionCAD:-ECCADProjOp( convert(E2,set),
  convert(A2,set), [u,v])) }; #convert(map(X->discrim(X,x),
  Ad),set);
```

$$A1 := \{v^2\} \tag{1.3.10}$$

Only one polynomial.

```
> cadL1:=ProjectionCAD:-CADFull( A1, [v], method=McCallum,
  output=listwithrep, finalCAD=OI): nops(%);
```

$$3 \tag{1.3.11}$$

```
> E2; factor(%);
```

$$[u^2 - v^2]$$

$$[(u - v)\ (u + v)] \tag{1.3.12}$$

NB nothing in E2 nullified at v=0.
There was no E1

```
> out := []:
  for i from 1 to nops(cadL1) do
    cell := cadL1[i]:
    stk := ProjectionCAD:-CADGenerateStack( cadL1[i], E2, [u,
  v], output=listwithrep):
    out := [op(out), op(stk)]:
  od:
  nops(out);
  cadL2:=out:
```

$$13 \tag{1.3.13}$$

Clearly E3 cannot be nullified by any (u,v) value independent of x

```
> E3;
```

$$[u^2 - v^2 + x + 1] \tag{1.3.14}$$

```
> out := []:
  for i from 1 to nops(cadL2) do
    cell := cadL2[i]:
    if cell[1][-1]::even then
      stk := ProjectionCAD:-CADGenerateStack( cell, E3, [x,u,
  v], output=listwithrep):
      out := [op(out), op(stk)]:
    else
      stk := ProjectionCAD:-CADGenerateStack( cell, [1], [x,u,
  v], output=listwithrep):
      out := [op(out), op(stk)]:
    fi:
  od:
  nops(out);
  cadL3:=out:
```

23 (1.3.15)

```
> out := []:
  for i from 1 to nops(cadL3) do
    cell := cadL3[i]:
    if cell[1][-2]::even then
      stk := ProjectionCAD:-CADGenerateStack( cell, E4, [y,x,
  u,v], output=listwithrep):
      out := [op(out), op(stk)]:
    else
      stk := ProjectionCAD:-CADGenerateStack( cell, [1], [y,x,
  u,v], output=listwithrep):
      out := [op(out), op(stk)]:
    fi:
  od:
  nops(out);
  cadL4:=out:
```

53 (1.3.16)

```
> out := []:
  for i from 1 to nops(cadL4) do
    cell := cadL4[i]:
    if cell[1][-1]::even and cell[1][-3]::even then
      stk := ProjectionCAD:-CADGenerateStack( cell, E5, [z,y,
  x,u,v], output=listwithrep):
      out := [op(out), op(stk)]:
    else
      stk := ProjectionCAD:-CADGenerateStack( cell, [1], [z,y,
  x,u,v], output=listwithrep):
      out := [op(out), op(stk)]:
    fi:
  od:
  nops(out);
  cadL5:=out:
```

113 (1.3.17)

```
> cadL5[13][2];
  cadL5[35][2];
  cadL5[58][2];
  cadL5[81][2];
  cadL5[103][2];
```

$$[v < 0,\ u = v,\ x = -1,\ y = 0,\ z = 1]$$
$$[v < 0,\ u = -v,\ x = -1,\ y = 0,\ z = 1]$$
$$[v = 0,\ u = 0,\ x = -1,\ y = 0,\ z = 1]$$
$$[0 < v,\ u = -v,\ x = -1,\ y = 0,\ z = 1]$$
$$[0 < v,\ u = v,\ x = -1,\ y = 0,\ z = 1] \qquad \textbf{(1.3.18)}$$

```
> Solution;
```

$$\{[x = -1,\ y = 0,\ z = 1,\ u = v],\ [x = -1,\ y = 0,\ z = 1,\ u = -v]\} \qquad \textbf{(1.3.19)}$$

Solution in 5 cells.

In the above we made choices about which EC to use.
We observe what would of happened with different choices.

# Different choices

```
> RunTheTest:=proc(Choice)
  local E5,E4,E3,E2,A5,A4,A3,A2,A1,cadL1,cadL2,cadL3,cadL4,
  cadL5,out,stk,cell,set,st,et,i:

  E5:={L1EC[Choice[1]]};
  E4:={L2EC[Choice[2]]};
  E3:={L3EC[Choice[3]]};
  E2:={L4EC[Choice[4]]};
  print(E5,E4,E3,E2);

  st:=time():

  A5:={f1,f2,f3,f4,g,h};

  A4 := ProjectionCAD:-ECCADProjOp( E5, A5, [z,y,x,u,v]):
  A4 := remove(X->X::constant, A4):
  A4 := ProjectionCAD:-PCAD_SFBasis(%, y);

  A3 := { op(ProjectionCAD:-ECCADProjOp( convert(E4,set),
  convert(A4,set), [y,x,u,v])), op(map(X->discrim(X,y), expand
  (A4)))}:
  A3 := remove(X->X::constant, A3):
  A3 := ProjectionCAD:-PCAD_SFBasis(%, x);

  A2 := { op(ProjectionCAD:-ECCADProjOp( convert(E3,set),
  convert(A3,set), [x,u,v])), op(map(X->discrim(X,x), A3)) }:
  A2 := remove(X->X::constant, A2):
  A2 := ProjectionCAD:-PCAD_SFBasis(%, u);

  A1 := { op(ProjectionCAD:-ECCADProjOp( convert(E2,set),
  convert(A2,set), [u,v]))}:
  A1 := remove(X->X::constant, A1);

  cadL1:=ProjectionCAD:-CADFull( A1, [v], method=McCallum,
  output=listwithrep, finalCAD=OI):
  #print("R1:", nops(%));

  out := []:
  for i from 1 to nops(cadL1) do
    cell := cadL1[i]:
    stk := ProjectionCAD:-CADGenerateStack( cadL1[i], E2, [u,
  v], output=listwithrep):
    out := [op(out), op(stk)]:
  od:
  cadL2:=out:
  #print("R2:", nops(%));
```

```
out := []:
for i from 1 to nops(cadL2) do
  cell := cadL2[i]:
  if cell[1][-1]::even then
    stk := ProjectionCAD:-CADGenerateStack( cell, E3, [x,u,
v], output=listwithrep):
    out := [op(out), op(stk)]:
  else
    stk := ProjectionCAD:-CADGenerateStack( cell, [1], [x,u,
v], output=listwithrep):
    out := [op(out), op(stk)]:
  fi:
od:
cadL3:=out:
#print("R3:", nops(%));

out := []:
for i from 1 to nops(cadL3) do
  cell := cadL3[i]:
  if cell[1][-2]::even then
    stk := ProjectionCAD:-CADGenerateStack( cell, E4, [y,x,
u,v], output=listwithrep):
    out := [op(out), op(stk)]:
  else
    stk := ProjectionCAD:-CADGenerateStack( cell, [1], [y,x,
u,v], output=listwithrep):
    out := [op(out), op(stk)]:
  fi:
od:
cadL4:=out:
#print("R4:", nops(%));

out := []:
for i from 1 to nops(cadL4) do
  cell := cadL4[i]:
  if cell[1][-1]::even and cell[1][-3]::even then
    stk := ProjectionCAD:-CADGenerateStack( cell, E5, [z,y,
x,u,v], output=listwithrep):
    out := [op(out), op(stk)]:
  else
    stk := ProjectionCAD:-CADGenerateStack( cell, [1], [z,y,
x,u,v], output=listwithrep):
    out := [op(out), op(stk)]:
  fi:
od:
cadL5:=out:
#print("R5:", nops(%));
```

```
  print("Cells:", nops(cadL5));
  et:=time()-st;
  print("Time:", et):

  return( nops(cadL5) );
  end proc:
```

```
> nops(L1EC), nops(L2EC), nops(L3EC), nops(L4EC);
```

$$4, 5, 3, 1 \tag{1.4.1}$$

```
> Choices := [
  [1,1,1,1], [1,1,2,1], [1,1,3,1],
  [1,2,1,1], [1,2,2,1], [1,2,3,1],
  [1,3,1,1], [1,3,2,1], [1,3,3,1],
  [1,4,1,1], [1,4,2,1], [1,4,3,1],
  [1,5,1,1], [1,5,2,1], [1,5,3,1],

  [2,1,1,1], [2,1,2,1], [2,1,3,1],
  [2,2,1,1], [2,2,2,1], [2,2,3,1],
  [2,3,1,1], [2,3,2,1], [2,3,3,1],
  [2,4,1,1], [2,4,2,1], [2,4,3,1],
  [2,5,1,1], [2,5,2,1], [2,5,3,1],

  [3,1,1,1], [3,1,2,1], [3,1,3,1],
  [3,2,1,1], [3,2,2,1], [3,2,3,1],
  [3,3,1,1], [3,3,2,1], [3,3,3,1],
  [3,4,1,1], [3,4,2,1], [3,4,3,1],
  [3,5,1,1], [3,5,2,1], [3,5,3,1],

  [4,1,1,1], [4,1,2,1], [4,1,3,1],
  [4,2,1,1], [4,2,2,1], [4,2,3,1],
  [4,3,1,1], [4,3,2,1], [4,3,3,1],
  [4,4,1,1], [4,4,2,1], [4,4,3,1],
  [4,5,1,1], [4,5,2,1], [4,5,3,1]
  ];
  nops(Choices);
```

$$Choices := [[1, 1, 1, 1], [1, 1, 2, 1], [1, 1, 3, 1], [1, 2, 1, 1], [1, 2, 2, 1], [1, 2, 3, 1], [1, 3, 1, 1], [1, 3, 2, 1], [1, 3, 3, 1], [1, 4, 1, 1], [1, 4, 2, 1], [1, 4, 3, 1], [1, 5, 1, 1], [1, 5, 2, 1], [1, 5, 3, 1], [2, 1, 1, 1], [2, 1, 2, 1], [2, 1, 3, 1], [2, 2, 1, 1], [2, 2, 2, 1], [2, 2, 3, 1], [2, 3, 1, 1], [2, 3, 2, 1], [2, 3, 3, 1], [2, 4, 1, 1], [2, 4, 2, 1], [2, 4, 3, 1], [2, 5, 1, 1], [2, 5, 2, 1], [2, 5, 3, 1], [3, 1, 1, 1], [3, 1, 2, 1], [3, 1, 3, 1], [3, 2, 1, 1], [3, 2, 2, 1], [3, 2, 3, 1], [3, 3, 1, 1], [3, 3, 2, 1], [3, 3, 3, 1], [3, 4, 1, 1], [3, 4, 2, 1], [3, 4, 3, 1], [3, 5, 1, 1], [3, 5, 2, 1], [3, 5, 3, 1], [4, 1, 1, 1], [4, 1, 2, 1], [4, 1, 3, 1], [4, 2, 1, 1], [4, 2, 2, 1], [4, 2, 3, 1], [4, 3, 1, 1], [4, 3, 2, 1], [4, 3, 3, 1], [4, 4, 1, 1], [4, 4, 2, 1], [4, 4, 3, 1], [4, 5, 1, 1], [4, 5, 2, 1], [4, 5, 3, 1]]$$

```
> Results:=[]:
  for ord in Choices do
  print("------------------");
  print(ord);
  N := RunTheTest(ord);
  Results:=[op(Results), N]:
  od:
```

"------------------"

[1, 1, 1, 1]

$\{z^2+x-y\}, \{4\,y^2\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 103

"Time:", 0.360

"------------------"

[1, 1, 2, 1]

$\{z^2+x-y\}, \{4\,y^2\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 103

"Time:", 0.346

"------------------"

[1, 1, 3, 1]

$\{z^2+x-y\}, \{4\,y^2\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 103

"Time:", 0.424

"------------------"

[1, 2, 1, 1]

$\{z^2+x-y\}, \{-u^2+v^2+x-y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.408

"------------------"

[1, 2, 2, 1]

$\{z^2+x-y\}, \{-u^2+v^2+x-y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.498

"------------------"

[1, 2, 3, 1]

$\{z^2+x-y\}, \{-u^2+v^2+x-y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.410

"------------------"

[1, 3, 1, 1]

$\{z^2+x-y\}, \{-u^2+v^2+x+y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.470

"----------"

[1, 3, 2, 1]

$\{z^2+x-y\}, \{-u^2+v^2+x+y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.352

"----------"

[1, 3, 3, 1]

$\{z^2+x-y\}, \{-u^2+v^2+x+y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.353

"----------"

[1, 4, 1, 1]

$\{z^2+x-y\}, \{u^2-v^2+x-y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.499

"----------"

[1, 4, 2, 1]

$\{z^2+x-y\}, \{u^2-v^2+x-y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.414

"----------"

[1, 4, 3, 1]

$\{z^2+x-y\}, \{u^2-v^2+x-y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.503

"----------"

[1, 5, 1, 1]

$\{z^2+x-y\}, \{u^2-v^2+x+y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.354

"----------"

[1, 5, 2, 1]

$\{z^2+x-y\}, \{u^2-v^2+x+y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.441

"----------"

[1, 5, 3, 1]

$\{z^2+x-y\}, \{u^2-v^2+x+y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.356

"----------"

[2, 1, 1, 1]

$\{-u^2+v^2+z^2-1\}, \{4\,y^2\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.483

"----------"

[2, 1, 2, 1]

$\{-u^2+v^2+z^2-1\}, \{4\,y^2\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.375

"----------"

[2, 1, 3, 1]

$\{-u^2+v^2+z^2-1\}, \{4\,y^2\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.467

"----------"

[2, 2, 1, 1]

$\{-u^2+v^2+z^2-1\}, \{-u^2+v^2+x-y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.406

"----------"

[2, 2, 2, 1]

$\{-u^2+v^2+z^2-1\}, \{-u^2+v^2+x-y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.501

"----------"

[2, 2, 3, 1]

$\{-u^2+v^2+z^2-1\}, \{-u^2+v^2+x-y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.408

"------------------"

[2, 3, 1, 1]

$\{-u^2+v^2+z^2-1\}, \{-u^2+v^2+x+y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.420

"------------------"

[2, 3, 2, 1]

$\{-u^2+v^2+z^2-1\}, \{-u^2+v^2+x+y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.494

"------------------"

[2, 3, 3, 1]

$\{-u^2+v^2+z^2-1\}, \{-u^2+v^2+x+y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.408

"------------------"

[2, 4, 1, 1]

$\{-u^2+v^2+z^2-1\}, \{u^2-v^2+x-y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.505

"------------------"

[2, 4, 2, 1]

$\{-u^2+v^2+z^2-1\}, \{u^2-v^2+x-y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.412

"------------------"

[2, 4, 3, 1]

$\{-u^2+v^2+z^2-1\}, \{u^2-v^2+x-y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.505

"------------------"

[2, 5, 1, 1]

$\{-u^2+v^2+z^2-1\}, \{u^2-v^2+x+y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.406

"------------------"

[2, 5, 2, 1]

$\{-u^2+v^2+z^2-1\}, \{u^2-v^2+x+y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.492

"----------"

[2, 5, 3, 1]

$\{-u^2+v^2+z^2-1\}, \{u^2-v^2+x+y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.406

"----------"

[3, 1, 1, 1]

$\{z^2+x+y\}, \{4\,y^2\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 103

"Time:", 0.429

"----------"

[3, 1, 2, 1]

$\{z^2+x+y\}, \{4\,y^2\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 103

"Time:", 0.337

"----------"

[3, 1, 3, 1]

$\{z^2+x+y\}, \{4\,y^2\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 103

"Time:", 0.423

"----------"

[3, 2, 1, 1]

$\{z^2+x+y\}, \{-u^2+v^2+x-y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.363

"----------"

[3, 2, 2, 1]

$\{z^2+x+y\}, \{-u^2+v^2+x-y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.356

"----------"

[3, 2, 3, 1]

$\{z^2+x+y\}, \{-u^2+v^2+x-y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.451

"----------"

[3, 3, 1, 1]

$\{z^2+x+y\}, \{-u^2+v^2+x+y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.426

"----------"

[3, 3, 2, 1]

$\{z^2+x+y\}, \{-u^2+v^2+x+y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.520

"----------"

[3, 3, 3, 1]

$\{z^2+x+y\}, \{-u^2+v^2+x+y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.410

"----------"

[3, 4, 1, 1]

$\{z^2+x+y\}, \{u^2-v^2+x-y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.449

"----------"

[3, 4, 2, 1]

$\{z^2+x+y\}, \{u^2-v^2+x-y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.355

"----------"

[3, 4, 3, 1]

$\{z^2+x+y\}, \{u^2-v^2+x-y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 93

"Time:", 0.366

"----------"

[3, 5, 1, 1]

$\{z^2+x+y\}, \{u^2-v^2+x+y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.500

"----------"

[3, 5, 2, 1]

$\{z^2+x+y\}, \{u^2-v^2+x+y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.412

"----------"

[3, 5, 3, 1]

$\{z^2+x+y\}, \{u^2-v^2+x+y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.516

"----------"

[4, 1, 1, 1]

$\{u^2-v^2+z^2-1\}, \{4y^2\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.383

"----------"

[4, 1, 2, 1]

$\{u^2-v^2+z^2-1\}, \{4y^2\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.481

"----------"

[4, 1, 3, 1]

$\{u^2-v^2+z^2-1\}, \{4y^2\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.376

"----------"

[4, 2, 1, 1]

$\{u^2-v^2+z^2-1\}, \{-u^2+v^2+x-y+1\}, \{x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.512

"----------"

[4, 2, 2, 1]

$\{u^2-v^2+z^2-1\}, \{-u^2+v^2+x-y+1\}, \{-u^2+v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.413

"----------"

[4, 2, 3, 1]

$\{u^2-v^2+z^2-1\}, \{-u^2+v^2+x-y+1\}, \{u^2-v^2+x+1\}, \{u^2-v^2\}$

"Cells:", 113

"Time:", 0.517

"----------"

[4, 3, 1, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{-u^2 + v^2 + x + y + 1\}, \{x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.430

"----------"

[4, 3, 2, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{-u^2 + v^2 + x + y + 1\}, \{-u^2 + v^2 + x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.506

"----------"

[4, 3, 3, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{-u^2 + v^2 + x + y + 1\}, \{u^2 - v^2 + x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.416

"----------"

[4, 4, 1, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{u^2 - v^2 + x - y + 1\}, \{x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.512

"----------"

[4, 4, 2, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{u^2 - v^2 + x - y + 1\}, \{-u^2 + v^2 + x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.423

"----------"

[4, 4, 3, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{u^2 - v^2 + x - y + 1\}, \{u^2 - v^2 + x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.512

"----------"

[4, 5, 1, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{u^2 - v^2 + x + y + 1\}, \{x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.404

"------------------"

[4, 5, 2, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{u^2 - v^2 + x + y + 1\}, \{-u^2 + v^2 + x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.525

"------------------"

[4, 5, 3, 1]

$\{u^2 - v^2 + z^2 - 1\}, \{u^2 - v^2 + x + y + 1\}, \{u^2 - v^2 + x + 1\}, \{u^2 - v^2\}$

"Cells:", 113

"Time:", 0.414 **(1.4.3)**

```
> Results;
```

[103, 103, 103, 113, 113, 113, 93, 93, 93, 113, 113, 113, 93, 93, 93, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 103, 103, 103, 93, 93, 93, 113, 113, 113, 93, 93, 93, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113, 113] **(1.4.4)**

```
> min(Results);
```

93 **(1.4.5)**

```
> for i from 1 to 60 do
  if Results[i]=93 then print(i, Choices[i]); fi:
  od:
```

7, [1, 3, 1, 1]

8, [1, 3, 2, 1]

9, [1, 3, 3, 1]

13, [1, 5, 1, 1]

14, [1, 5, 2, 1]

15, [1, 5, 3, 1]

34, [3, 2, 1, 1]

35, [3, 2, 2, 1]

36, [3, 2, 3, 1]

40, [3, 4, 1, 1]

41, [3, 4, 2, 1]

42, [3, 4, 3, 1] **(1.4.6)**