```
> restart;
```

We are using the standard Maple 16 Library

```
> libname:="/home/me350/Programs/Maple16/lib":
```

We make use of the following packages:

```
> with(RegularChains):
  with(SemiAlgebraicSetTools):
```

Additionally, we use Maple code written at the University of Bath: The ProjectionCAD package (should be hosted alongside this worksheet).

```
> read("ProjectionCAD.mpl"):
  with(ProjectionCAD):
```

"This is V3.18 of the ProjectionCAD module from 11th February 2015, designed and tested for use in Maple 18."

Throughout the paper we focussed on some worked examples to demonstrate our ideas. We demonstrate how the cell counts reported were obtained.

Section 1.3

In this section we introduced the following polynomials

```
> f1:=x^2+y^2-1;
  g1:=x*y-1/4;
  f2:=(x-4)^2+(y-1)^2-1;
  g2:=(x-4)*(y-1)-1/4;
```

$$f1 := y^2 + x^2 - 1$$

$$g1 := y\,x - \frac{1}{4}$$

$$f2 := (x-4)^2 + (y-1)^2 - 1$$

$$g2 := (x-4)\,(y-1) - \frac{1}{4}$$

We use variable ordering y>x

```
> vars:=[y,x]:
  R:=PolynomialRing(vars):
```

The standard sign-invariant CAD procedure in Maple builds a CAD with 317 cells (as does Qepcad - see qepcad folder).

```
> CylindricalAlgebraicDecompose([f1,g1,f2,g2], R, output=list):
  nops(%);
```

317

Qepcad can also use the implict EC to build a CAD with 249 cells.

Section 3.3

We again use

```
> f1:=x^2+y^2-1;
  g1:=x*y-1/4;
  f2:=(x-4)^2+(y-1)^2-1;
  g2:=(x-4)*(y-1)-1/4;
```

$$f1 := y^2 + x^2 - 1$$

$$g1 := y\,x - \frac{1}{4}$$

$$f2 := (x-4)^2 + (y-1)^2 - 1$$

$$g2 := (x-4)\,(y-1) - \frac{1}{4}$$

```
> vars:=[y,x]:
  R:=PolynomialRing(vars):
```

We calculate projection sets, induced CADs of the real line and CADs of the plane for the worked example Phi under different projection operators.

First, using McCallum's sign-invariant operator P(A):

```
> CADProjection( [f1,g1,f2,g2], vars, method=McCallum):
  remove(X->X in [f1,f2,f3,f4], %);
```

$$\{x, x-5, x-4, x-3, x-1, x+1, 4\,y\,x-1, x^2-4\,x+1, 68\,x^2-272\,x+285, 16\,x^4-16\,x^2+1, 4\,y\,x-16\,y-4\,x+15, 16\,x^4-256\,x^3+1520\,x^2-3968\,x+3841, 16\,x^4-128\,x^3+256\,x^2-8\,x+1, 16\,x^4-128\,x^3+256\,x^2+8\,x-31, y^2+x^2-2\,y-8\,x+16\}$$

```
> CADFull( [f1,g1,f2,g2], vars, method=McCallum, retcad=1,
  output=list): nops(%);
```

$$41$$

```
> CADFull( [f1,g1,f2,g2], vars, method=McCallum, output=list):
  nops(%);
```

$$317$$

Second with McCallum's operator for an EC (implicit in this case) P[E](A):

```
> ECCADProjFactors( [f1*f2, [f1,f2,g1,g2]], vars):
  remove(X->X in [f1,f2,f3,f4], %);
```

$$\left\{x-5, x-3, x-1, x+1, y\,x-\frac{1}{4}, x^2-4\,x+\frac{285}{68}, x^4-x^2+\frac{1}{16}, y\,x-x-4\,y+\frac{15}{4}, x^4-16\,x^3+95\,x^2-248\,x+\frac{3841}{16}, x^4-8\,x^3+16\,x^2-\frac{1}{2}\,x+\frac{1}{16}, x^4-8\,x^3+16\,x^2+\frac{1}{2}\,x-\frac{31}{16}, y^2+x^2-2\,y-8\,x+16\right\}$$

```
> ECCAD( [f1*f2, [f1,f2,g1,g2]], vars, retcad=1, output=list):
  nops(%);
```

33

```
> ECCAD( [f1*f2, [f1,f2,g1,g2]], vars, output=list): nops(%);
```

145

Qepcad's approach using this projection gives 249. The difference is actually in the lifting (see Section x.x).

Third with the new TTICAD operator.

```
> TTICADProjFactors( [ [f1,[g1]], [f2,[g2]] ], vars):
  remove(X->X in [f1,f2,f3,f4], %);
```

$$\left\{x-5, x-3, x-1, x+1, y\,x-\frac{1}{4}, x^2-4\,x+\frac{285}{68}, x^4-x^2+\frac{1}{16}, y\,x-x-4\,y+\frac{15}{4}, x^4-16\,x^3+95\,x^2-248\,x+\frac{3841}{16}, y^2+x^2-2\,y-8\,x+16\right\}$$

```
> TTICAD( [ [f1,[g1]], [f2,[g2]] ], vars, retcad=1, output=list):
  nops(%);
```

25

```
> TTICAD( [ [f1,[g1]], [f2,[g2]] ], vars, output=list): nops(%);
```

105

We also consider Psi. The sign-invariant approach would be the same while the EC approach is no longer valid. The TTICAD approach, however, still gives savings.

```
> TTICADProjFactors( [ [f1,[g1]], [[],[f2,g2]] ], vars):
  remove(X->X in [f1,f2,f3,f4], %);
```

$$\left\{x-5, x-4, x-3, x-1, x+1, y\,x-\frac{1}{4}, x^2-4\,x+\frac{285}{68}, x^4-x^2+\frac{1}{16}, y\,x-x-4\,y+\frac{15}{4}, x^4-16\,x^3+95\,x^2-248\,x+\frac{3841}{16}, x^4-8\,x^3+16\,x^2+\frac{1}{2}\,x-\frac{31}{16}, y^2+x^2-2\,y-8\,x+16\right\}$$

```
> TTICAD( [ [f1,[g1]], [[],[f2,g2]] ], vars, retcad=1, output=
  list): nops(%);
```

31

```
> TTICAD( [ [f1,[g1]], [[],[f2,g2]] ], vars, output=list): nops
  (%);
```

183

Example 22

```
> vars:=[w,z,y,x]:
  R:=PolynomialRing(vars):
> f := x + y + z + w;
  g := z*y - x^2*w;
```

$$f := x + y + z + w$$

$$g := -x^2 w + z y$$

```
> CADFull([f,g], vars, method=McCallum, output=list): nops(%);
```

557

```
> TTICAD( [[f,[g]]], vars, output=list): nops(%);
```

165

The example has only one clause and so the projection theory is just that of McCallum's 1999 paper.

The point of the example is to show that our improved lifting avoids theoretical failure from non-well-orientedness.
Compare with Qepcad which produces 221 cells but also an error message warning against the validity of the output.

Example 23

```
> f1:=x^2+y^2-1; g1:=x*y-1/4;
```

$$f1 := y^2 + x^2 - 1$$

$$g1 := y x - \frac{1}{4}$$

```
> vars:=[y,x]:
  R:=PolynomialRing(vars):
```

SI CAD of real line has 15 cells, i.e. identifies 7 points.

```
> CADFull([f1,g1], vars, method=McCallum, retcad=1, output=list):
  nops(%);
```

15

SI CAD of plane:

```
> CADFull([f1,g1], vars, method=McCallum, output=list): nops(%);
```

83

Using the EC:

```
> ECCAD([f1,[g1]], vars, output=list): nops(%);
```

53

Look at the cell divisions. For x<-1 there is no splitting according to g1 (which Qepcad does)

```
> ECCAD([f1,[g1]], vars, output=piecewise);
```

$$[regular\_chain, [[-2, -2], [0, 0]]]$$

$$\begin{cases} [regular\_chain, [[-1, -1], [-1, -1]]] & y < 0 \\ [regular\_chain, [[-1, -1], [0, 0]]] & y = 0 \\ [regular\_chain, [[-1, -1], [1, 1]]] & 0 < y \end{cases}$$

$$\begin{cases} \left[regular\_chain, \left[\left[-\frac{63}{64}, -\frac{63}{64}\right], [-2, -2]\right]\right] & y < -\sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{63}{64}, -\frac{63}{64}\right], \left[-\frac{1}{4}, -\frac{1}{8}\right]\right]\right] & y = -\sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{63}{64}, -\frac{63}{64}\right], \left[-\frac{1}{16}, -\frac{1}{16}\right]\right]\right] & -\sqrt{-x^2+1} < y < \sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{63}{64}, -\frac{63}{64}\right], \left[0, \frac{1}{4}\right]\right]\right] & y = \sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{63}{64}, -\frac{63}{64}\right], [2, 2]\right]\right] & \sqrt{-x^2+1} < y \end{cases}$$

$$\begin{cases} \left[regular\_chain, \left[\left[-\frac{31}{32}, -\frac{123}{128}\right], [-2, -2]\right]\right] & y < -\sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{31}{32}, -\frac{123}{128}\right], \left[-\frac{3}{8}, -\frac{1}{8}\right]\right]\right] & y = -\sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{31}{32}, -\frac{123}{128}\right], [0, 0]\right]\right] & -\sqrt{-x^2+1} < y < \sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{31}{32}, -\frac{123}{128}\right], \left[\frac{1}{8}, \frac{3}{8}\right]\right]\right] & y = \sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{31}{32}, -\frac{123}{128}\right], [2, 2]\right]\right] & \sqrt{-x^2+1} < y \end{cases}$$

$$\begin{cases} \left[regular\_chain, \left[\left[-\frac{313}{512}, -\frac{313}{512}\right], [-2, -2]\right]\right] & y < -\sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{313}{512}, -\frac{313}{512}\right], \left[-1, -\frac{3}{4}\right]\right]\right] & y = -\sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{313}{512}, -\frac{313}{512}\right], [0, 0]\right]\right] & -\sqrt{-x^2+1} < y < \sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{313}{512}, -\frac{313}{512}\right], \left[\frac{3}{4}, 1\right]\right]\right] & y = \sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{313}{512}, -\frac{313}{512}\right], [2, 2]\right]\right] & \sqrt{-x^2+1} < y \end{cases} \quad RootOf(16\_Z$$

$$\begin{cases} \left[regular\_chain, \left[\left[-\frac{67}{256}, -\frac{33}{128}\right], [-2, -2]\right]\right] & y < -\sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{67}{256}, -\frac{33}{128}\right], \left[-1, -\frac{7}{8}\right]\right]\right] & y = -\sqrt{-x^2+1} \\ \left[regular\_chain, \left[\left[-\frac{67}{256}, -\frac{33}{128}\right], [0, 0]\right]\right] & -\sqrt{-x^2+1} < y < \sqrt{-x^2+1} \end{cases}$$

Example 24

```
> f1:=x^2+y^2-1;
  g1:=x*y-1/4;
  f2:=(x-4)^2+(y-1)^2-1;
  g2:=(x-4)*(y-1)-1/4;
```

$$f1 := y^2 + x^2 - 1$$

$$g1 := y\,x - \frac{1}{4}$$

$$f2 := (x-4)^2 + (y-1)^2 - 1$$

$$g2 := (x-4)\,(y-1) - \frac{1}{4}$$

```
> vars:=[y,x]:
  R:=PolynomialRing(vars):
> ECCAD( [f1*f2,[ f1,f2,g1,g2]], vars): nops(%);
```

$$145$$

Example 25

```
> f1:=x^2+y^2+z^2-1;
  g1:=x*y*z-1/4;
  f2:=(x-4)^2+(y-1)^2+(z-2)^2-1;
  g2:=(x-4)*(y-1)*(z-2)-1/4;
```

$$f1 := x^2 + y^2 + z^2 - 1$$

$$g1 := x\,y\,z - \frac{1}{4}$$

$$f2 := (x-4)^2 + (y-1)^2 + (z-2)^2 - 1$$

$$g2 := (x-4)\,(y-1)\,(z-2) - \frac{1}{4}$$

```
> vars:=[z,y,x]:
  R:=PolynomialRing(vars):

> TTICAD( [[f1,[g1]], [f2,[g2]]], [z,y,x], output=list): nops(%);
```

$$109$$

```
> ECCAD( [f1*f2, [f1,f2,g1,g2]], [z,y,x], output=list): nops(%);
```

$$353$$

Example 32

```
> f := z + y*w;
  g := y*x + 1;
  h := w*(z+1) + 1;
```

$$f := y\,w + z$$
$$g := y\,x + 1$$
$$h := w\,(z + 1) + 1$$

```
> vars:=[w,z,y,x]:
  R:=PolynomialRing(vars):
> CADR4 := TTICAD( [ [f,[g,h]] ], vars, output=listwithrep): nops
  (%);
```

467

```
> CADR3 := TTICAD( [ [f,[g,h]] ], vars, output=listwithrep,
  retcad=3): nops(%);
```

169

Note that f is nullified when y=z=0. I.e. on these 5 cells:

```
> select(X->X[2][2..2]=[y=0], CADR3):
  select(X->X[2][3..3]=[z=0], %); nops(%);
```

$[[[1, 4, 4], [x < 0, y = 0, z = 0], [\mathit{regular\_chain}, [[-1, -1], [0, 0], [0, 0]]]], [[2, 4, 4], [x = 0, y = 0, z = 0], [\mathit{regular\_chain}, [[0, 0], [0, 0], [0, 0]]]], [[3, 6, 4], [0 < x < 4, y = 0, z = 0], [\mathit{regular\_chain}, [[2, 2], [0, 0], [0, 0]]]], [[4, 4, 4], [x = 4, y = 0, z = 0], [\mathit{regular\_chain}, [[4, 4], [0, 0], [0, 0]]]], [[5, 6, 4], [4 < x, y = 0, z = 0], [\mathit{regular\_chain}, [[5, 5], [0, 0], [0, 0]]]]]$

5

The lifting set varies from cell to cell:

```
> select(X->X[1][1..3]=[2,4,4], CADR4);
```

$[[[2, 4, 4, 1], [x = 0, y = 0, z = 0, w < -1], [\mathit{regular\_chain}, [[0, 0], [0, 0], [0, 0], [-2, -2]]]], [[2, 4, 4, 2], [x = 0, y = 0, z = 0, w = -1], [\mathit{regular\_chain}, [[0, 0], [0, 0], [0, 0], [-1, -1]]]], [[2, 4, 4, 3], [x = 0, y = 0, z = 0, -1 < w], [\mathit{regular\_chain}, [[0, 0], [0, 0], [0, 0], [0, 0]]]]]$

```
> select(X->X[1][1..3]=[2,4,1], CADR4);
```

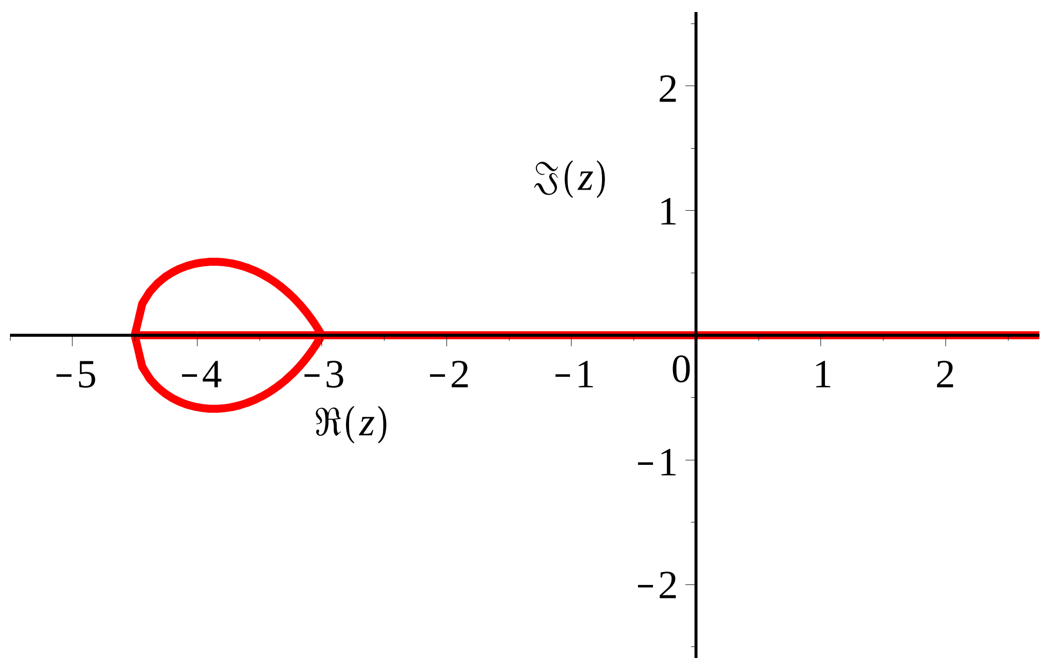
$[[[2, 4, 1, 1], [x = 0, y = 0, z < -1, w = w], [\mathit{regular\_chain}, [[0, 0], [0, 0], [-2, -2], [0, 0]]]]]$

Example 33

```
> Kahan:=2*arccosh(1+2*z/3)-arccosh( (5*z+12)/(3*(z+4)) ) = 2*
  arccosh( 2*(z+3)*sqrt( (z+3)/(27*(z+4))));
```

$$Kahan := 2\operatorname{arccosh}\left(1+\frac{2z}{3}\right)-\operatorname{arccosh}\left(\frac{5z+12}{3z+12}\right)=2\operatorname{arccosh}\left(2(z+3)\sqrt{\frac{z+3}{27z+108}}\right)$$

```
> FAout := FunctionAdvisor(branch_cuts, lhs(Kahan) - rhs(Kahan),
  plot=2d, title="", color=red, thickness=3);
```



$$FAout := \left[2\operatorname{arccosh}\left(1+\frac{2z}{3}\right)-\operatorname{arccosh}\left(\frac{5z+12}{3z+12}\right)-2\operatorname{arccosh}\left(2(z+3)\sqrt{\frac{z+3}{27z+108}}\right),\right.$$
$$-4<z,\ -\frac{9}{2}<z<-4,\ \Im(z)=\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\;(\Re(z)+3)}{2\,\Re(z)+5}\ \mathbf{And}\ -\frac{9}{4}$$
$$<\Re(z),\ \Im(z)=-\frac{\sqrt{-4\,\Re(z)^2-28\,\Re(z)-45}\;(\Re(z)+3)}{2\,\Re(z)+5}\ \mathbf{And}\ -\frac{9}{4}<\Re(z),\ \Im(z)$$

$$= \frac{\sqrt{-4\,\Re(z)^2 - 28\,\Re(z) - 45}\;(\Re(z)+3)}{2\,\Re(z)+5} \textbf{ And } \Re(z) \le -3 \textbf{ And } -\frac{9}{2} < \Re(z), \Im(z)$$

$$= \frac{\sqrt{-4\,\Re(z)^2 - 28\,\Re(z) - 45}\;(\Re(z)+3)}{2\,\Re(z)+5} \textbf{ And } -\frac{5}{2} < \Re(z) \textbf{ And } \Re(z) < -\frac{9}{4}, \Im(z) =$$

$$-\frac{\sqrt{-4\,\Re(z)^2 - 28\,\Re(z) - 45}\;(\Re(z)+3)}{2\,\Re(z)+5} \textbf{ And } \Re(z) \le -3 \textbf{ And } -\frac{9}{2} < \Re(z), \Im(z) =$$

$$-\frac{\sqrt{-4\,\Re(z)^2 - 28\,\Re(z) - 45}\;(\Re(z)+3)}{2\,\Re(z)+5} \textbf{ And } -\frac{5}{2} < \Re(z) \textbf{ And } \Re(z) < -\frac{9}{4} \Big]$$

We turn these descriptions of branch cuts into 7 pairs of equations and inequalities

```
> BC := [[4*y*(2*x^3+2*x*y^2+21*x^2+5*y^2+72*x+81), [-4*x^4+4*y^4
  -52*x^3+12*x*y^2-225*x^2+63*y^2-324*x]], [2*y, [2*x+9]], [8*y,
  [8*x^2+8*y^2+56*x+96]], [y, [x^2+y^2+7*x+12]], [4*y*(2*x^3+2*x*
  y^2+21*x^2+5*y^2+72*x+81), [-4*x^4+4*y^4-52*x^3+12*x*y^2-252*
  x^2+36*y^2-540*x-432]], [4*y*(2*x^3+2*x*y^2+21*x^2+5*y^2+72*
  x+81), [4*x^4-4*y^4+52*x^3-12*x*y^2+225*x^2-63*y^2+324*x]], [2*
  y, [-6-2*x]], [2*y, [2*x]], [8*y, [-8*x^2-8*y^2-56*x-96]], [8*
  y, [2*x^2+2*y^2+8*x]]];
```

$$BC := [[4\,y\,(2\,x^3 + 2\,y^2\,x + 21\,x^2 + 5\,y^2 + 72\,x + 81), [-4\,x^4 + 4\,y^4 - 52\,x^3 + 12\,y^2\,x - 225\,x^2 + 63\,y^2 - 324\,x]], [2\,y, [2\,x+9]], [8\,y, [8\,x^2 + 8\,y^2 + 56\,x + 96]], [y, [x^2 + y^2 + 7\,x + 12]], [4\,y\,(2\,x^3 + 2\,y^2\,x + 21\,x^2 + 5\,y^2 + 72\,x + 81), [-4\,x^4 + 4\,y^4 - 52\,x^3 + 12\,y^2\,x - 252\,x^2 + 36\,y^2 - 540\,x - 432]], [4\,y\,(2\,x^3 + 2\,y^2\,x + 21\,x^2 + 5\,y^2 + 72\,x + 81), [4\,x^4 - 4\,y^4 + 52\,x^3 - 12\,y^2\,x + 225\,x^2 - 63\,y^2 + 324\,x]], [2\,y, [-6 - 2\,x]], [2\,y, [2\,x]], [8\,y, [-8\,x^2 - 8\,y^2 - 56\,x - 96]], [8\,y, [2\,x^2 + 2\,y^2 + 8\,x]]]$$

```
> F := map(X->op([op(1,X),op(op(2,X))]),  BC);
```

$$F := [4\,y\,(2\,x^3 + 2\,y^2\,x + 21\,x^2 + 5\,y^2 + 72\,x + 81), -4\,x^4 + 4\,y^4 - 52\,x^3 + 12\,y^2\,x - 225\,x^2 + 63\,y^2 - 324\,x, 2\,y, 2\,x + 9, 8\,y, 8\,x^2 + 8\,y^2 + 56\,x + 96, y, x^2 + y^2 + 7\,x + 12, 4\,y\,(2\,x^3 + 2\,y^2\,x + 21\,x^2 + 5\,y^2 + 72\,x + 81), -4\,x^4 + 4\,y^4 - 52\,x^3 + 12\,y^2\,x - 252\,x^2 + 36\,y^2 - 540\,x - 432, 4\,y\,(2\,x^3 + 2\,y^2\,x + 21\,x^2 + 5\,y^2 + 72\,x + 81), 4\,x^4 - 4\,y^4 + 52\,x^3 - 12\,y^2\,x + 225\,x^2 - 63\,y^2 + 324\,x, 2\,y, -6 - 2\,x, 2\,y, 2\,x, 8\,y, -8\,x^2 - 8\,y^2 - 56\,x - 96, 8\,y, 2\,x^2 + 2\,y^2 + 8\,x]$$

```
> CADFull(F, [y,x], method=McCallum): nops(%);
```

409

```
> TTICAD(BC, [y,x]): nops(%);
```

55

```
> CADFull(F, [x,y], method=McCallum): nops(%);
```

1143

```
> TTICAD(BC, [x,y]): nops(%);
```

39

Example 34

```
> f1 := (y-1) - x^3 + x^2 + x;
  f2 := (-y-1) - x^3 + x^2 + x;
  g1 := y - x/4 + 1/2;
  g2 := -y - x/4 + 1/2;
```

$$f1 := -x^3 + x^2 + x + y - 1$$

$$f2 := -x^3 + x^2 + x - y - 1$$

$$g1 := y - \frac{x}{4} + \frac{1}{2}$$

$$g2 := -y - \frac{x}{4} + \frac{1}{2}$$

```
> ECCAD( [f1, [f2,g1,g2]], [y,x]): nops(%);
  ECCAD( [f2, [f1,g1,g2]], [y,x]): nops(%);
```

$$39$$

$$39$$

```
> TTICAD( [ [f1,[g1]], [f2,[g2]] ], [y,x]): nops(%);
```

$$31$$